Stability of the equilibrium point of the centre of mass of an extensible cable connected satellites system in case of circular orbit in three dimensional motions.

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Abstract

In a linear motion of a system of two cable-connected satellites, one stable equilibrium point must exist when perturbative forces like solar pressure, shadow of the earth due to solar pressure, air resistance magnetic force and oblateness of the earth act simultaneously. Actually, we have obtained one stable point of equilibrium in case of perturbative forces like the shadow of the earth due to solar radian pressure and oblateness of the earth act together on the three dimensional motion in case of circular motion of two cable connected satellites. Liapunov's theorem has been used to examine the stability of the equilibrium point

Key words: Circular Motion, Equilibrium Points, Extensible Cable Connected satellites, Liapunov Theorem, Langrage's Equations, Solar Radiation Pressure, Stability, Oblateness of Earth.

1. Introduction

The present paper is concerned with the stability of the equilibrium point of the centre of mass of the system in case of three dimensional motion in circular orbit under the influence of the shadow of the earth due to solar pressure in the central gravitational field of oblate earth. Beletsky, V.V is the pioneer worker in this field. This paper is an attempt towards the generalisation of works done by him.

2. Equations of motion

The equations of motion of one of the two satellites moving along Keplerian elliptic orbit under the influence of the shadow of the earth due to solar pressure and oblateness of the earth in Nechvill's coordinates system may be obtained by exploiting Lagrange's equations of motion of first kind in the form:

$$x''-2y'-3x\rho-\frac{4Bx}{\rho}+A\rho^3\,\Psi_1\cos\in\cos(\nu-\alpha)=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_o}{\rho\sqrt{x^2+y^2+z^2}}\right]x$$

$$y''+2x'+\frac{By}{\rho}-A\rho^3\,\Psi_1\cos\in\sin(\nu-\alpha)=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_o}{\rho\sqrt{x^2+y^2+z^2}}\right]y$$

and $$z''+z+\frac{Bz}{\rho}-A\rho^3\,\Psi_1\sin\in=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_o}{\rho\sqrt{x^2+y^2+z^2}}\right]z \qquad (1)$$

Where,

$A=\dfrac{P^3}{\mu}\left(\dfrac{B_1}{m_1}-\dfrac{B_2}{m_2}\right)$ = Solar pressure parameter

$B=\dfrac{3k_2}{P^2}$ = oblateness force parameter.

Ψ_1 = Shadow function parameter.

$$\overline{\lambda_\alpha} = \frac{P^3}{\mu}\lambda_\alpha = \frac{p^3}{\mu l_o}\left(\frac{m_1+m_2}{m_1 m_2}\right)\lambda$$

m_1 and m_2 being masses of the two satellites and l_0 is the natural length of the cable connecting two satellites. Here dashes denote differentiation w.r.to true anomaly v.

The condition of constraint is given by,

$$x^2 + y^2 + z^2 \le \frac{{l_0}^2}{\rho^2} \tag{2}$$

For the circular orbit of the centre of mass of the system, we have $e=0$ Hence $\rho = \frac{1}{1+e\cos\nu} = 1$ and $\rho^1 = 0$.

Putting $\rho\ =1$ and $\rho' = 0$ in (1), we get the equations of motion in the from

$$x'' - 2y' - (3+4B)x + A\Psi_1 \cos\in\cos(\nu-\alpha) = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]x$$

$$y'' + 2x' + By - A\Psi_1 \cos\in\sin(\nu-\alpha) = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]y$$

$$and \quad z'' + (1+B)z - A\Psi_1 \sin\in = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]z \tag{3}$$

Where, $r = \sqrt{x^2+y^2+z^2}$

Condition of constraint (2) reduces to

$$x^2 + y^2 + z^2 \le {l_0}^2 \tag{4}$$

Let us assume that in case of circular orbit, the true anomaly ν for the elliptic orbit will be replaced by τ whose value is as follow.

$$\tau = \omega_0 t \tag{5}$$

Where ω_0 is the angular velocity of the centre of mass of the system in case of circular orbit and t is the time.

Hence (3) can be rewritten as

$$x'' - 2y' - (3+4B)x + A\Psi_1 \cos\in\cos(\tau-\alpha) = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]x$$

$$y'' + 2x' + By - A\Psi_1 \cos\in\sin(\tau-\alpha) = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]y$$

$$and \ z'' + (1+B)z - A\Psi_1 \sin\in = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]z \tag{6}$$

If the inequality sign holds in (4), then the motion takes place with loose string and the motion in called free motion otherwise the motion is called constrained motion.

To find the Jacobi's integral, we use the following averaged values of periodic terms in (6) as

$$\frac{1}{2\pi}\left[\int_{\substack{-\theta \\ \Psi_1=0}}^{\theta} A\psi_1\cos\in\cos(\tau-\alpha)d\tau + \int_{\substack{\theta \\ \Psi_1=1}}^{2\pi-\theta} A\psi_1\cos\in\cos(\tau-\alpha)d\tau\right] = \frac{-A\cos\in\cos\alpha\sin\theta}{\pi}$$

$$and\ \frac{1}{2\pi}\left[\int_{\substack{-\theta \\ \Psi_1=0}}^{\theta} A\psi_1 \cos\in\sin(\tau-\alpha)d\tau + \int_{\substack{\theta \\ \Psi_1=1}}^{2\pi-\theta} A\psi_1 \cos\in\sin(\tau-\alpha)d\tau\right] = \frac{A\cos\in Sin\alpha\sin\theta}{\pi} \tag{7}$$

Where θ is taken to be constant.

Thus, the equations of motion given by (6) are being described by using average values given in (7) in the form:-

$$x''-2y'-(3+4B)x-\frac{A\cos\in\cos\alpha\sin\theta}{\pi} = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]x$$

$$y''+2x'+By-\frac{A\cos\in Sin\alpha\sin\theta}{\pi} = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]y$$

$$and\ z''+(1+B)z-A\sin\in = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right]z\,. \tag{8}$$

We see that the equations of motion given by (8) do not contain time t explicitly. Hence there must exist a Jacobian integral of the problem.

Multiplying the three equations of (8) by $2x'$, $2y'$ and $2z'$ respectively and adding them together, we get after integrating Jacobi's integral as

$$x'^2+y'^2+z'^2-(3+4B)x^2+By^2+(1+B)z^2-\frac{1}{\pi}2Ax\cos\in\cos\alpha\sin\theta$$

$$\frac{2Ay\cos\in\sin\alpha\sin\theta}{\pi}-2Az\sin\in+\overline{\lambda_\alpha}\left[x^2+y^2+z^2-2l_0\sqrt{x^2+y^2+z^2}\right]=h\,. \tag{9}$$

Where h is the constant of integration.

The surface of zero velocity can be obtained by putting $x'=y'=z'=0$ in (9) in the form

$$(3+4B)x^2-By^2-(1+B)z^2+\frac{1}{\pi}2Ax\cos\in\cos\alpha\sin\theta+\frac{1}{\pi}2Ay\cos\in\sin\alpha\sin\theta$$

$$+2Az\sin\in-\overline{\lambda_\alpha}\left[x^2+y^2+z^2-2l_0\sqrt{x^2+y^2+z^2}\right]+h=0 \tag{10}$$

Hence, we conclude that the satellite of mass m_1 will move inside the boundaries of different curves of zero velocity represented by (10) for different values of Jacobian constant h.

3. Equilibrium point of the problem

We have obtained the system of equations given by (8) for the motion of the system in rotating frame of reference. It has been assumed that the system is moving with effective constraints and the cable connecting the two satellites will remain tight.

The equilibrium point of the system is given by the constant values of the coordinates in the rotating frame of reference.

Now, let $x = x_0$, $y = y_0$ and $z = z_0$ give the equilibrium position

When x_0, y_0 and z_0 are constant.

Hence $x' = 0 = y' = z'$

$$x'' = 0 = y'' = z''$$

Thus, equations given by (8) take the form:

$$-(3+4B)x_0 - \frac{A}{\pi}\cos\in\cos\alpha\sin\theta = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r_0}\right]x_0$$

$$By_0 - \frac{A}{\pi}\cos\in\sin\alpha\sin\theta = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r_0}\right]y_0$$

$$(1+B)z_0 - A\sin\in = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r_0}\right]z_0 \tag{11}$$

Where $r_0 = \sqrt{x_0^{\,2} + y_0^{\,2} + z_0^{\,2}}$

From (11) it follows that it is very difficult to get the solution of in its present form. Hence we are compelled to make the following assumptions; $\alpha = 0$, $\in = 0$ putting $\alpha = 0$ & $\in = 0$ in (11) we get

$$-(3+4B)x_0 - \frac{A}{\pi}\sin\theta = -\overline{\lambda_\alpha}\left[1-\frac{\lambda_0}{r_0}\right]x_0$$

$$By_0 = -\overline{\lambda_\alpha}\left[1-\frac{\lambda_0}{r_0}\right]y_0$$

$$(1+B)z_0 = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r_0}\right]z_0 \tag{12}$$

From (12), we get three equilibrium points out of these three equilibrium point, only one equilibrium point given by

$$\left[\frac{\overline{\lambda_\alpha}l_0\pi + A\sin\theta}{\pi\left(\overline{\lambda_\alpha} - 3 - 4B\right)}, 0, 0\right] \tag{13}$$

gives us a meaningful value of Hook's modulus of elasticity λ, if $0 < \theta < 90^0$.

4. Stability of the system

We shall examine the stability of the equilibrium point of the system given by (13) in the sense of Liapunov. For this,

$$Let\ x = a = \frac{\overline{\lambda_\alpha}l_0\pi + A\sin\theta}{\pi\left(\overline{\lambda_\alpha} - 3 - 4B\right)}, y = b = 0 \ \ and \quad z = c = 0$$

Let us assume that there are small variation in the coordinate at the given equilibrium position [a, b, c]. Let η_1, η_2 and η_3 denote small variations in the coordinates for the position of equilibrium. Thus we get

$$x = a + \eta_1 \qquad y = o + \eta_2 \qquad z = o + \eta_2$$

$$\therefore x' = \eta_1^{'} \qquad y' = \eta_2^{'} \qquad z' = \eta_3' \tag{14}$$

$$x'' = \eta_1^{''} \qquad y'' = \eta_2^{''} \qquad z'' = \eta_2^{''}$$

Hence on putting $\alpha = 0$ and $\in = 0$ in (8) and using (14), we get the variational equations of motion for the system in the form:

$$\eta_1^{''} - 2\eta_2^{'} - (3+4B)(a+\eta_1) - \frac{A\sin\theta}{\pi} = -\overline{\lambda_\alpha}\left[1-\frac{l_0}{r}\right](a+\eta_1)$$

$$\eta_2'' + 2\eta_1' + B\eta_2 = -\overline{\lambda_\alpha}\left[1 - \frac{l_0}{r}\right]\eta_2$$

$$\eta_3'' + (1+B)\eta_3 = -\overline{\lambda_\alpha}\left[1 - \frac{l_0}{r}\right]\eta_3. \qquad (15)$$

Where $r^2 = (a+\eta_1)^2 + \eta_2{}^2 + \eta_3{}^2$

Multiplying the three equations of (15) by $2(a+\eta_1)'$, $2\eta'_2$ and $2\eta'_3$ respectively and adding them together and then integrating, we get Jacobian integral in the form:

$$\eta_1^2 + \eta_2^2 + \eta_3^2 - (3+4B)(a+\eta_1)^2 + (1+B)\eta_3^2 - \frac{2A\eta_1 \sin\theta}{\pi} + B\eta_2^2$$

$$+ \overline{\lambda_\alpha}\left[(a+\eta_1)^2 + \eta_2{}^2 + \eta_3{}^2\right] - 2\overline{\lambda_\alpha}\ell_0\left[(a+\eta_1)^2 + \eta_2{}^2 + \eta_3{}^2\right]^{\frac{1}{2}} = h \qquad (16)$$

Where h is the constant of integration.

To test the stability in the sense of Liapunov, We take Jacobi's integral given by (16) as Liapunov's function $\nu(\eta_1', \eta_2', \eta_3', \eta_1, \eta_2, \eta_3)$ and is obtained by expanding the terms of (16) as

$$\nu(\eta_1', \eta_2', \eta_3'\eta_1, \eta_2, \eta_3) = \eta_1'^2 + \eta_2'^2 + \eta_3'^2 + \eta_1^2\left[\overline{\lambda_\alpha} - 3 + 4B\right] + \eta_2^2\left[-\frac{\overline{\lambda_\alpha} l_0}{a}\right] + B$$

$$+ \eta_3{}^2\left[1 + B + \overline{\lambda_\alpha} - \frac{\overline{\lambda_\alpha} l_0}{a}\right]$$

$$+ \eta_1\left[-2(3+4B)a - \frac{2A\sin\theta}{\pi} + 2a\overline{\lambda_\alpha}\,a - 2\overline{\lambda_\alpha} l_0\right]$$

$$+ a\left[-(3+4B)a + \overline{\lambda_\alpha}\,(a - 2\ell_0)\right] + 0(3) = h \qquad (17)$$

Where 0(3) stands for third and higher order terms in small quantities η_1, η_2 and η_3

By Liapunov's theorem on stability it follows that the only criterion for given equilibrium position (a, o, o) to be stable is that ν defined by (17) must be positive definite and for this the following four conditions must be satisfied:

(i) $\quad -2(3+4B)a - \dfrac{2A\sin\theta}{\pi} + 2\overline{\lambda}_\alpha a - 2\overline{\lambda}_\alpha l_0 = 0$

(ii) $\quad \overline{\lambda_\alpha} - 3 - 4B > 0$ $\qquad (18)$

(iii) $\quad \overline{\lambda_\alpha} - \dfrac{\overline{\lambda_\alpha} l_0}{a} > 0$

and (iv) $\quad 1 + B + \overline{\lambda_\alpha} - \dfrac{\overline{\lambda_\alpha} l_0}{a} > 0$

In order to have clear picture of the stability, we have to show that all the four conditions mentioned in (18) must be satisfied identically.

Condition (i) putting $a = \dfrac{\overline{\lambda_\alpha}\,\pi\, l_0 + A\sin\theta}{\pi(\overline{\lambda_\alpha} - 3 - 4B)}$, in condition (i) of (18)

We find that

L.H.S. $= 2\left[\overline{\lambda_\alpha} - 3 - 4B\right]\left[\dfrac{\overline{\lambda_\alpha}\,\pi\, l_0 + A\sin\theta}{\pi\left(\overline{\lambda_\alpha} - 3 - 4B\right)}\right] - \dfrac{2}{\pi}\left[\overline{\lambda_\alpha}\,\pi\, l_0 + A\sin\theta\right] = 0$

Hence the condition (i) is identically satisfied.

Condition (ii) since (a, o, o) is the equilibrium point

So $\quad a = \dfrac{\overline{\lambda_\alpha}\,\pi\, l_0 + A\sin\theta}{\pi\left(\overline{\lambda_\alpha} - 3 - 4B\right)} > 0$

But $\overline{\lambda_\alpha}\,\pi\, l_0 + A\sin\theta > 0 \quad as\ \overline{\lambda_\alpha} > 0, \pi > 0, l_0 > 0, A > 0\ \ and\ 0 < 0 < \pi/2$

Hence denominator of $a = \overline{\lambda_\alpha} - 3 - 4B > 0$

$\therefore$ Condition (ii) is identically satisfied

L.H.S. of (iv) $1 + B + \overline{\lambda_\alpha}\left(1 - \dfrac{\lambda_0}{a}\right) = 1 + B + \left[1 - \dfrac{\lambda_0\left(\pi\overline{\lambda_\alpha} - 3\pi - 4B\pi\right)}{\overline{\lambda_\alpha}\,\lambda_0\pi + A\cos\in\sin\theta}\right]\overline{\lambda_\alpha}$

Condition (iii)

$\overline{\lambda_\alpha} - \dfrac{\overline{\lambda_\alpha} l_0}{a} = \overline{\lambda_\alpha}\left[1 - \dfrac{l_0}{a}\right] = \overline{\lambda_\alpha}\left[1 - \dfrac{l_0\left(\overline{\lambda_\alpha}\pi - 3\pi - 4B\pi\right)}{\overline{\lambda_\alpha}\,\pi l_0 + A\sin\theta}\right] = \dfrac{\pi l_0\overline{\lambda_\alpha}(3 + 4B)}{\overline{\lambda_\alpha}\,\pi l_0 + A\sin\theta}$ >0, as $0 < \theta < \pi/2$

$\therefore$ 3rd condition is identically satisfied

Using condition (iii) we find that

Condition (iv) $1 + B + \overline{\lambda_\alpha}\left(1 - \dfrac{l_0}{a}\right) > 0$

Hence Fourth condition is also identically satisfied. Thus, we see that all the four conditions of (18) for stability of the equilibrium point (a, 0, 0) are identically satisfied.

Hence, we find that the equilibrium point (a, o, o) given by (13) is stable in the sense of Liapunov.

Conclusion: we conclude that the equilibrium position [a ,0, 0] of the system is stable in the sense of Liapunov.

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